Using Lines to Model Relationships

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What is a Mathematical Model?

Brainstorm: Let’s write 10 words that come to your mind when you hear the words “Mathematical Model.”

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A *mathematical model is a mathematical equation, formula, or a graph that presents the relationship between two or more variables.*

Class Activity

The gas stations in the U.S. dispense gas at the rate of 10 gallons per minute.

a) When you arrived at the gas station, you had 5 gallons of gas in your car. You pumped gas for 30 seconds (1/2 minute) and stopped. How much gas did you have in your tank when you stopped pumping gas?

b) Assume that the rate at which the gas is being pumped is fixed at 10 gallons per minute. Moreover, assume that you have 5 gallons of gas in your tank before beginning to pump gas. Consider the following two variables:

t: the amount of time (in minutes) that you pump gas
g: the amount of gas in your tank after t minutes

Write a formula that relates g to t, using the above information.

c) Assume that the rate at which the gas is being pumped is fixed at 10 gallons per minute. Moreover, assume that you have b gallons of gas in your tank before beginning to pump gas. Consider the following two variables:

t: the amount of time (in minutes) that you pump gas
g: the amount of gas in your tank after t minutes

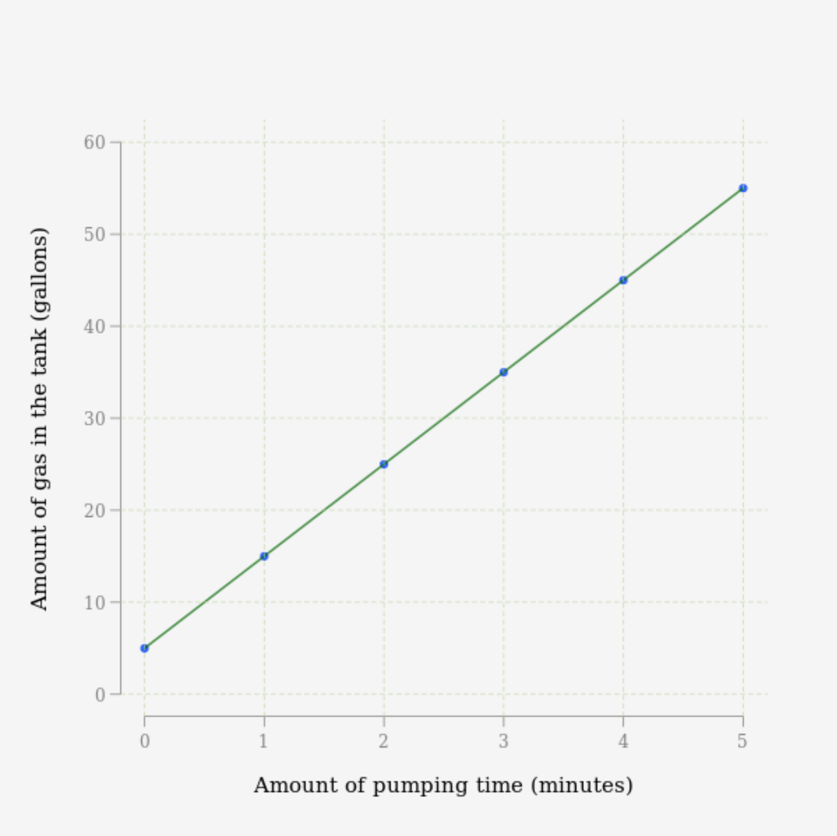
Write a formula that relates g to t, using the above information.

Mathematical Model: Pumping Gas

$$\underbrace{g}_{\substack{\text{Amount at}\\ \text{time } t\\ \text{(gallons)}}} = \underbrace{b}_{\substack{\text{Initial}\\ \text{amount}\\ \text{(gallons)}}} + \underbrace{10}_{\substack{\text{Dispense}\\ \text{Rate}\\ \text{(gallons/ min.)}}} \times \underbrace{t}_{\substack{\text{Pumping}\\ \text{time}\\ \text{(Minutes)}}}$$

Graphical Presentation of the Model

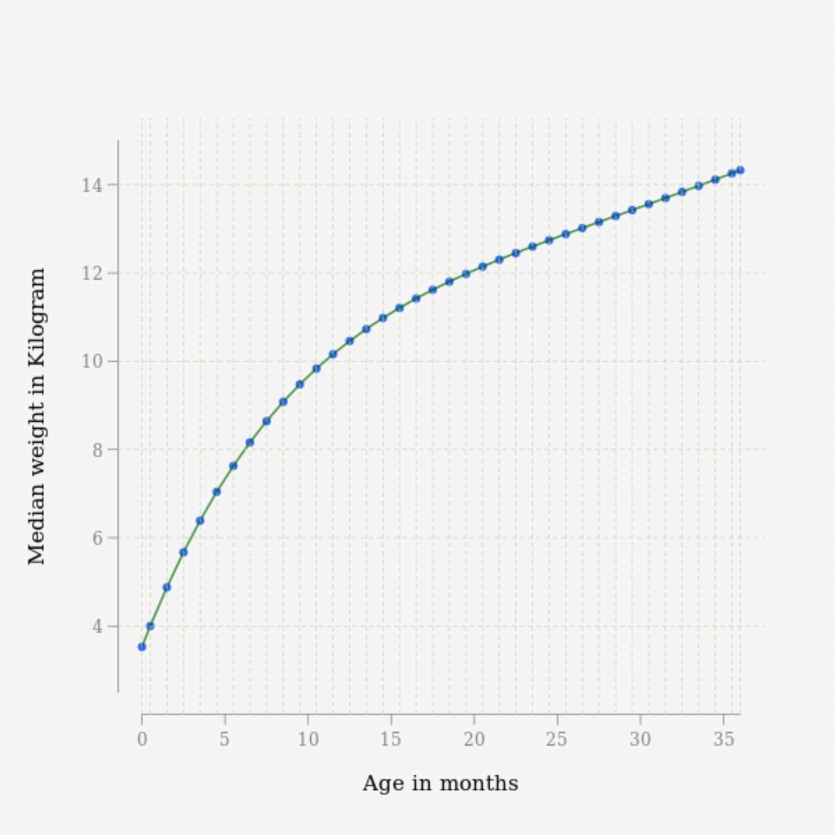
The amount of gas in the tank after t minutes, assuming that we initially have 5 gallons of gas in the car.



Mathematical Model: Age and Weight of Infants

A mathematical relationship between age of infants (in months) and their weight (in Kilogram)

Not all relationships are not summarized by a line!



Mathematical Model & Measuring Change

A mathematical model can be used to study how changes in one variable (the predictor) affect changes in another variable (the response).

Terminology & Notation

Predictor $\rightarrow$ x

Response $\rightarrow$ y

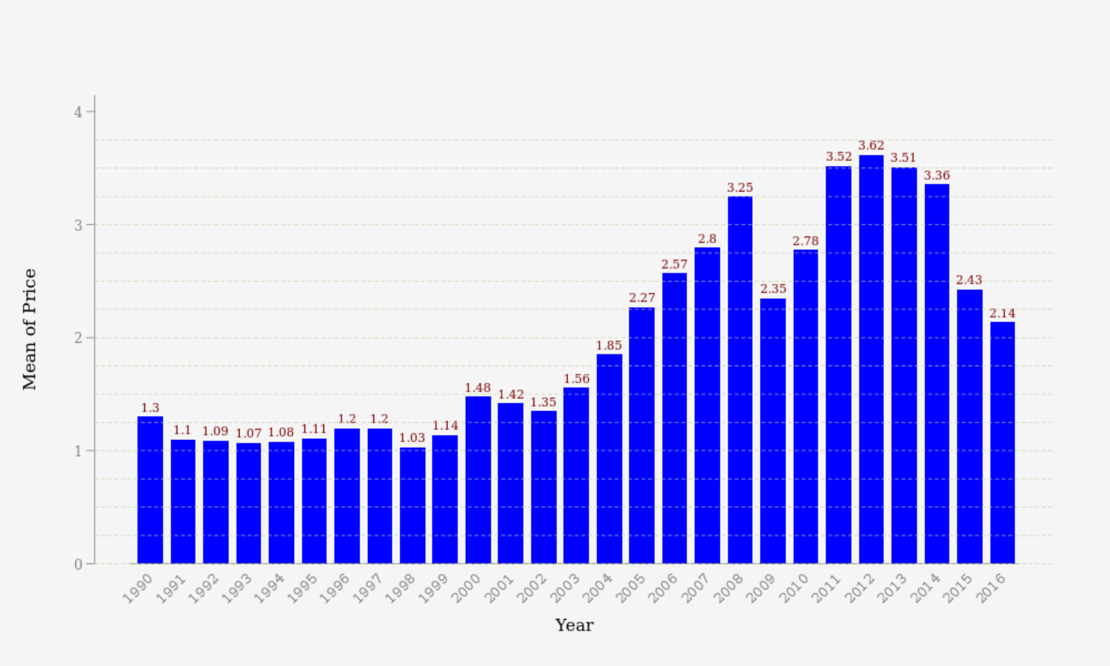
Change and Absolute Change

Class activity:

Consider the data given in the following graph:

a) How much lower or higher was the price of gas in 2016 as compared to 2000?

b) How much lower or higher was the price of gas in 2000 as compared to 2016?



Change and Absolute Change

If x_1 is a reference value for a variable and x_2 is another measure of the same variable, then

Change from x_1 (reference value) to $x_2 = \Delta x = x_2 - x_1$

Solution:

If reference date is the year 2000: $\Delta x = 2.14 - 1.48 = 0.66$

If reference date is the year 2016: $\Delta x = 1.48 - 2.14 = -0.66$

Change and Absolute Change

$$\Delta x > 0 \;\; \rightarrow Increase\ in\ value$$

$$\Delta x < 0 \;\; \rightarrow Decrease\ in\ value$$

Change, Absolute Change

The magnitude of the difference in values of a variable is referred to as the absolute change.

$$\text{Absolute Change} = |\Delta x| = |x_2 - x_1| = |x_1 - x_2|$$

Absolute change in average price of gas from 2000 to 2016 is $0.66 per gallon.

Relative Change

From the years 1990 to 2016, did the average car price increase more than the average gas price?

	Average Price in Dollars	
Year	Car	Gasoline
1990	15,045	1.30
2016	25,332	2.14

Change in car prices:

$$\Delta x = x_2 - x_1 = 25{,}332 - 15{,}045 = 10{,}287$$

Change in gas prices:

$$\Delta x = x_2 - x_1 = 2.14 - 1.30 = 0.84$$

Is this comparison informative?

Relative Change

To make a meaningful comparison of change in values of two variables whose values differ in order of magnitude, we compare their relative change.

If x_1 is a reference value for a variable and x_2 is another measure of the same variable, then

Relative Change from x_1 (reference value) to $x_2 = \frac{\Delta x}{x_1} = \frac{x_2 - x_1}{x_1}$

Relative Change

What is the relative change in the average car prices from the years 1990 to 2016?

Relative change (reference year 1990) $= \frac{x_2 - x_1}{x_1} = \frac{25{,}332 - 15{,}045}{15{,}045} \approx 0.68$

Percentage Change = Relative Change $\times\ 100$

The average price of cars has increased by 68% from the year 1990 to the year 2016.

Class Activity: Relative Change

a) What is the relative change in the average gas prices from the years 1990 to 2016?

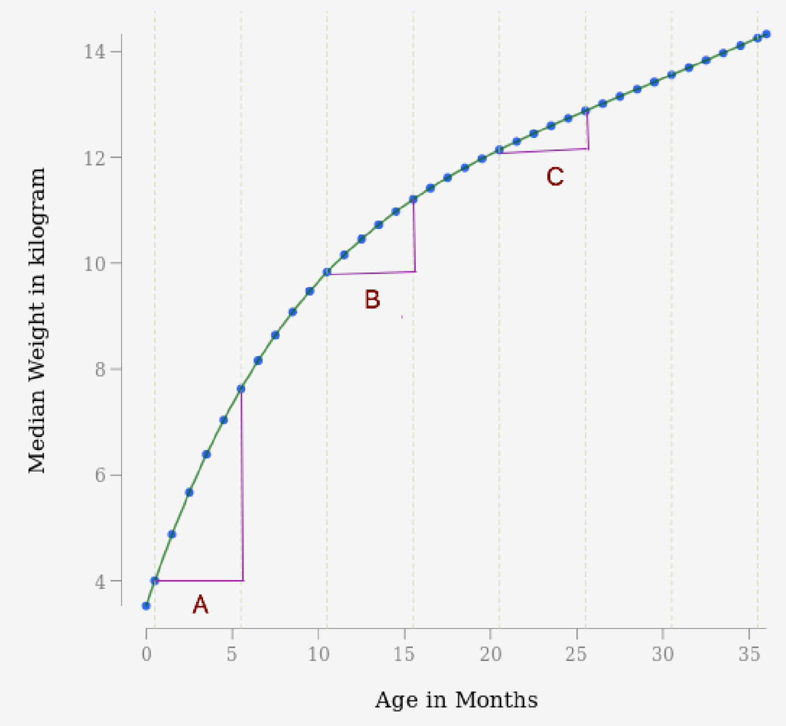
b) What is the percentage change in the average gas prices from the years 1990 to 2016?

c) How does the percentage change in average price of cars compare to that of gasoline from the years 1990 to 2016?

Average Rate of Change

Average rate of change is used to calibrate change in one variable as another variable changes over a range of its values.

The graph highlights changes in weight of infants in three five-month intervals:
A: 0.5 to 5.5 months
B: 10.5 to 15.5 months
C: 20.5 to 25.5 months
In which of the five month periods (A, B, or C) the change in weight is the most?



Average Rate of Change

Suppose that values of a variable x change from a reference value x_1 to x_2. Moreover, suppose that the values of another variable y at x_1 and x_2 are y_1 and y_2, respectively. Then, *the average rate of change* in y as x changes from x_1 to x_2 is

$$\text{Average rate of change in } y \text{ in the interval } [x_1, x_2] = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Average Rate of Change

Infant Weight Data:

Age in Months (x)	0.5	5.5	10.5	15.5	20.5	25.5
Weight in Kilogram (y)	4.0	7.6	9.8	11.2	12.1	12.9

Average rate of change in weight from 0.5 months to 5.5 months =

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7.6 - 4.0}{5.5 - 0.5} = \frac{3.6}{5} = 0.72 \text{ Kilograms per month}$$

Average Rate of Change

Class Activity: Obtain the average rate of change in the weight of infants over each of the following periods:

a) From 10.5 to 15.5 months

b) From 20.5 to 25.5 months

c) From 0.5 to 25.5 months

Average Rate of Change

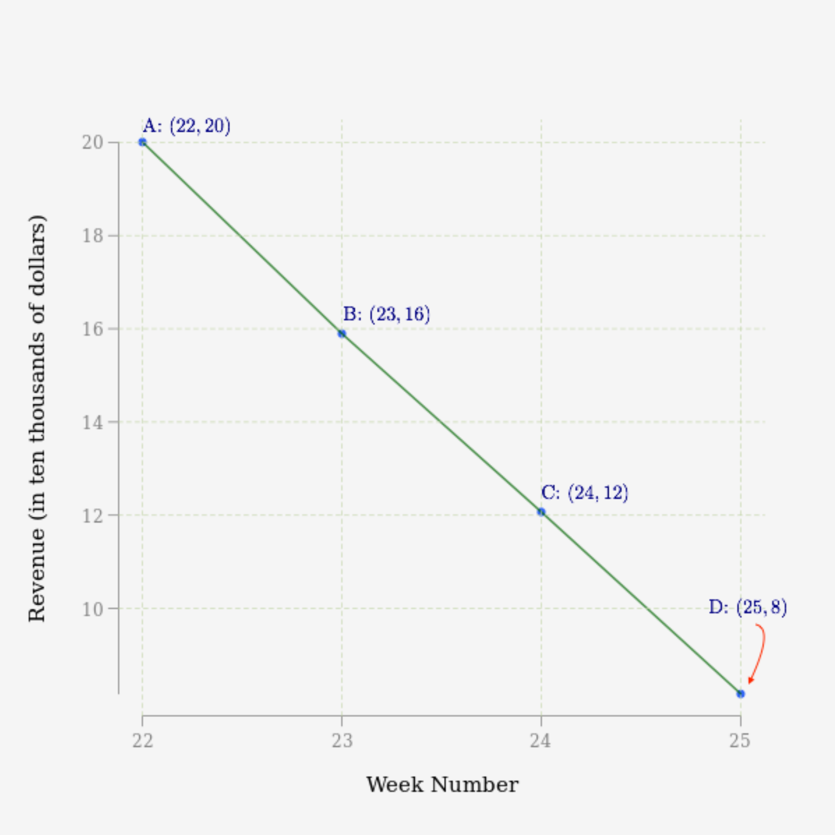
Class Activity: The graph on the right shows the revenue for the Harry Potter movie in weeks 22, 23, 24, and 25 after its release in 2001. Obtain the average rate of the change in the revenue y for the following periods x.

a) From weeks 22 to 23

b) From weeks 22 to 24

c) From weeks 23 to 25

d) From weeks 22 to 25



Linear and Nonlinear Relationship

When the graph relating two variables is a straight line, the average rate of change between any two points is constant. In this case we say there is a linear relationship between x and y.

When the graph relating two variables does not follow a straight line, the average rate of change differs in at least two intervals of x. In this case we say there is a nonlinear relationship between x and y.

(Average) Rate of Change and Slope of a Line

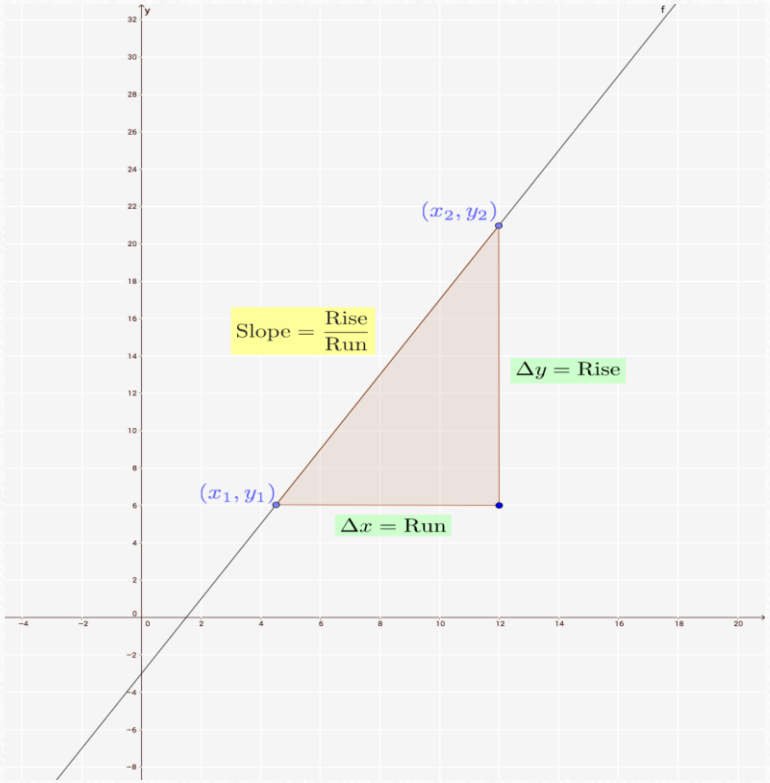
The average *rate of change* between any two points on a line is called the *slope* of the line. Or, in other words, the slope of a line is the rate of change in y per unit change in x.

Slope is commonly denoted by m.

Slope; rise over run

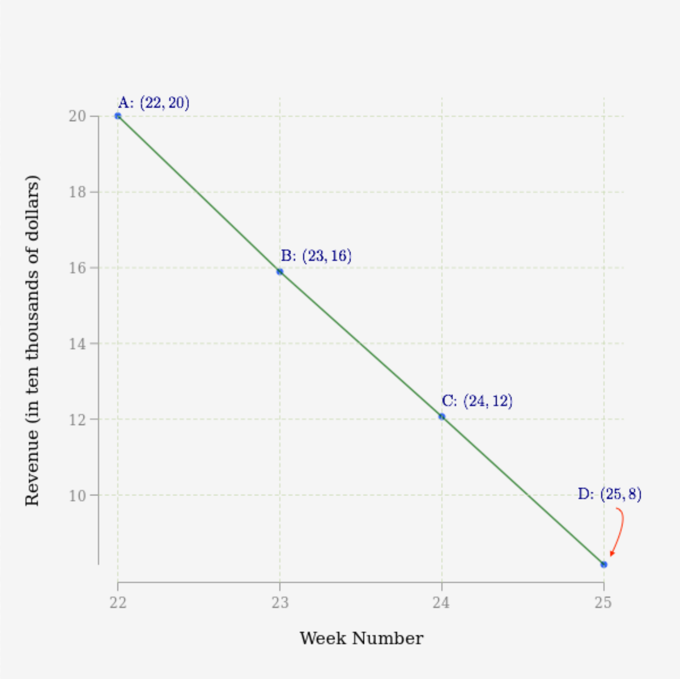
Given a reference point (x_1, y_1) and another point (x_2, y_2) on the same line, the slope of the line is calculated as "*rise over run*." Specifically,

$$\boxed{\text{Slope} = m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.}$$



Slope: The Harry Potter Revenue Data

Class activity: Compute the slope of the line for the Harry Potter data and interpret the slope.



Poll Question: Forms for Equation of a Line

Which of these forms would you introduce first?

a) Slope-intercept form: $y = mx + b$

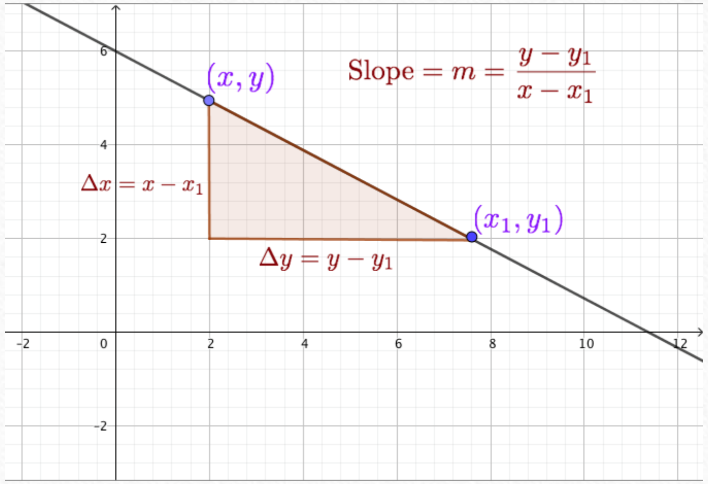
b) Point-slope form: $y - y_1 = m(x - x_1)$

c) Standard form: $ax + by = c$

Equation of a Line

Since for a line, the rate of change (slope) is a constant value over any interval of x, then for any reference point (x_1, y_1) and any other point (x, y) on the same line, we have

$$m = \frac{y - y_1}{x - x_1}.$$



This leads to the *point-slope formula* for a line:

$$y - y_1 = m(x - x_1)$$

Applying the Point Slope Formula

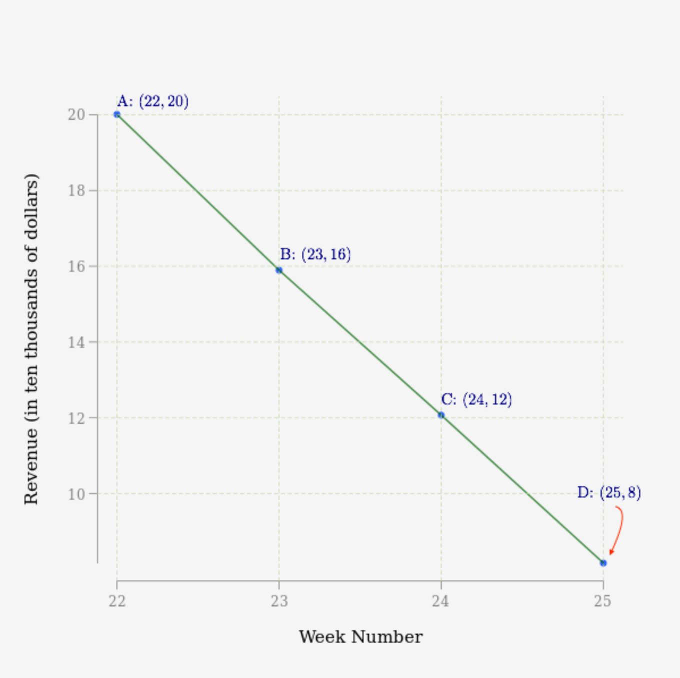
Apply the point slope formula to obtain the equation of the line for the Harry Potter revenue data for weeks 22 to 25.

Step 1: Choose any two points to obtain the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 20}{25 - 22} = -4$$

Step 2: Apply the point-slope formula.

$$\underbrace{y - 20}_{y_1} = \underbrace{(-4)}_{m}(x - \underbrace{22}_{x_1})$$



The Slope-Intercept Formula for a Line

Using algebra, the point-slope form of the line can be converted to another form of equation of a line called the *slope-intercept* form.

$$\boxed{\; y = \underset{\substack{\downarrow \\ \text{Slope}}}{m}\, x + \underset{\substack{\downarrow \\ y\text{-intercept}}}{b} \;}$$

Slope-Intercept form: Harry Potter

Write the Harry Potter equation in the slope-intercept form.

$$y = \underbrace{20}_{y_1} + (\underbrace{-4}_{m})(x - \underbrace{22}_{x_1}) \underbrace{\Longrightarrow}_{\substack{\text{Distribute } -4 \\ \text{over } (x-22)}} y = 20 + (-4)x + (-4)(-22)$$

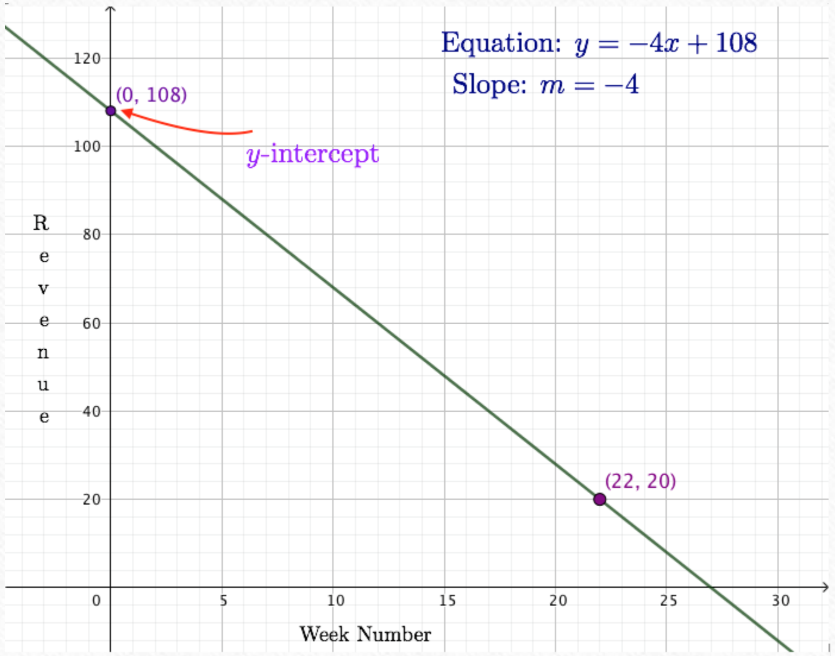
$$\underbrace{\Longrightarrow}_{\substack{\text{Collect and add} \\ \text{constant terms}}} y = -4x + 108.$$

Vertical-Intercept

In the slope-intercept form $y = mx + b$ the value b is the *vertical-intercept*, also called the *y-intercept*.

The vertical intercept is the value of y at $x = 0$.

The y-intercept is interpretable, if the value of $x = 0$ has a meaning in the context of the problem.



The Horizontal Intercept

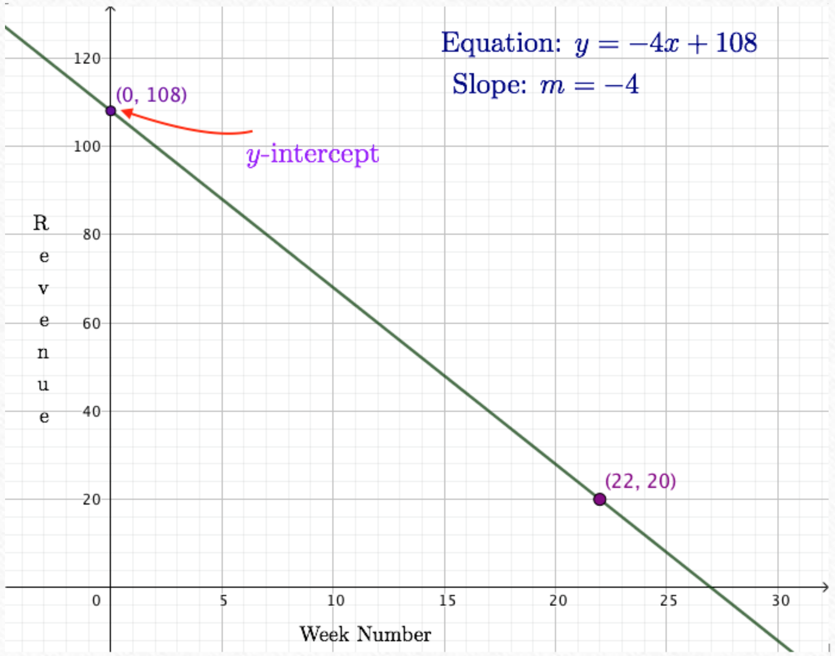
The horizontal intercept, or x-intercept is the value of x at $y = 0$.

Class Activity:

a) Set $y = 0$ in the Harry-Potter equation and solve for x. This will give you the horizontal intercept.

b) Locate the horizontal intercept on the graph.

c) Interpret the horizontal intercept.



Comparing Lines – Intersection of two lines

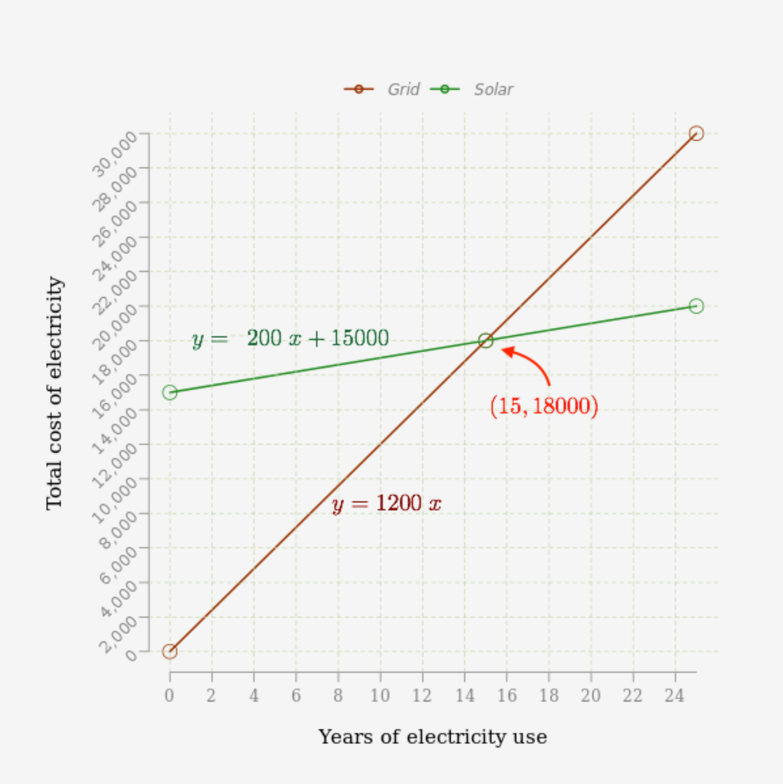
Class Activity: Electricity Cost

Public Grid: $1200 per year

Solar: $15,000 installation cost plus $200 per year.

a) After how many years would the cost of public grid be equal to the cost of installing solar energy?

b) Would you invest in solar energy, if you plan to sell your house in 10 years?

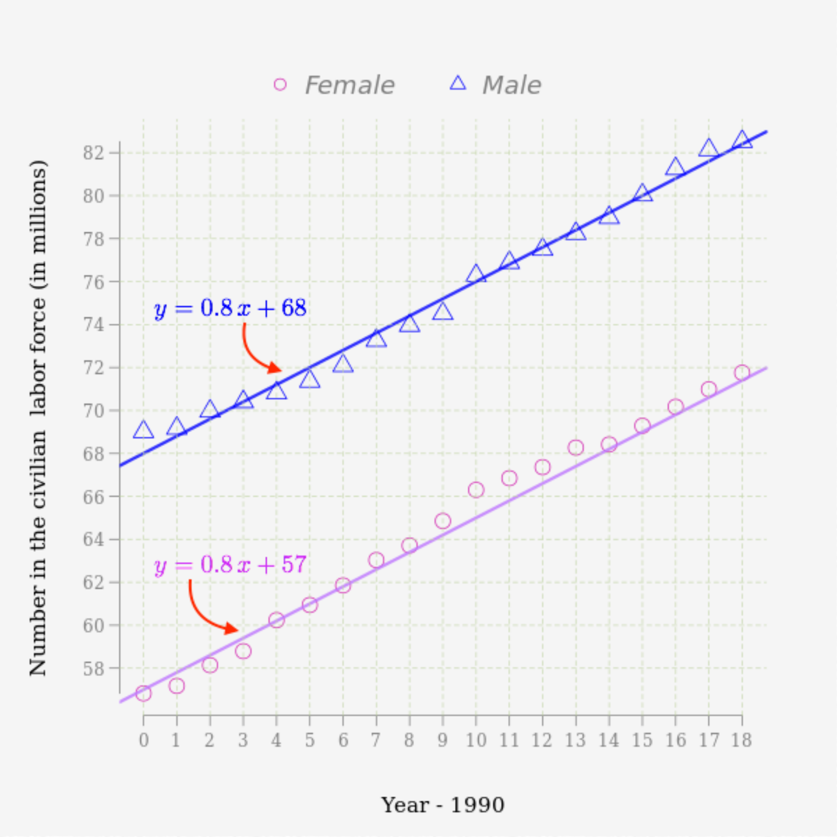


Comparing lines – Parallel lines

The graph shows the numbers in the civilian work force on the U.S. between the years 1990 and 2008 for male and females.

a) Is the rate of change (slope) for the number in the workforce the same for both male and females?

b) Interpret the vertical-intercept for male and female in the context of the problem.



Poll Question: Technology Use

What software/technology, if any, are you using/planning to use in your introductory statistics course?